

# Russian Doll Renormalization Group and Superconductivity

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We show that an extension of the standard BCS Hamiltonian leads to an infinite number of condensates with different energy gaps and self-similar properties, described by a cyclic RG flow of the BCS coupling constant which returns to its original value after a finite RG time.

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The Renormalization Group (RG) continues to be one of the most important tools for studying the qualitative and quantitative properties of quantum field theories and many-body problems in Condensed Matter physics. The emphasis so far has been mainly on flows toward fixed points in the UV or IR. Recently, an entirely novel kind of RG flow has been discovered in a number of systems wherein the RG exhibits a cyclic behavior: after a *finite* RG transformation the couplings return to their original values and the cycle repeats itself. Thus if one decreases the size of the system by a specific factor that depends on the coupling constants, one recovers the initial system, much like a Russian doll, or quantum version of the Mandelbrot set. Bedaque, Hammer and Van Kolck observed this behavior in a 3-body hamiltonian of interest in nuclear physics [1]. This motivated Glazek and Wilson to define a very simple quantum-mechanical hamiltonian with similar properties [2]. In the meantime such behavior was proposed for a certain regime of anisotropic current-current interactions in 2 dimensional quantum field theory [3].

The models in [1, 2] are problems in zero-dimensional quantum mechanics, and are thus considerably simpler than the quantum field theory in [3]. In the latter, standard quantum field theory methods of the renormalization group were used, however knowledge of the beta function to all orders was necessary to observe the cyclic flow. What is somewhat surprising is that the model considered in [3] is not very exotic, and is in fact a well-known theory that arises in many physical problems: at one-loop it is nothing more than the famous Kosterlitz-Thouless RG flow, where the cyclic regime corresponds to  $|g_{\perp}| > |g_{\parallel}|$ . This motivated us to find a simpler many-body problem that captures the essential features of the cyclic RG behavior. We found that a simple extension of the BCS hamiltonian has the desirable properties. Namely, our model is based on the BCS hamiltonian with scattering potential  $V_{jj'}$  equal to  $g + i\theta$  for  $\varepsilon_j > \varepsilon_{j'}$  and  $g - i\theta$  for  $\varepsilon_j < \varepsilon_{j'}$  in units of the energy spacing  $\delta$ .

The main features of the spectrum are the following. For large system size, there are an infinite number of BCS condensates, each characterized by an energy gap  $\Delta_n$  which depends on  $g, \theta$ . The role of these many condensates becomes clearer when we investigate the RG properties. As in the models considered in [1, 2, 3], the

RG flow possesses jumps from  $g = +\infty$  to  $g = -\infty$  and a new cycle begins. Let  $L = e^{-s}L_0$  denote the RG scale, which in our problem corresponds to  $N$  the number of unperturbed energy levels, and  $\lambda$  the period of an RG cycle:  $g(e^{-\lambda}L) = g(L)$ . We show that  $\lambda = \pi/\theta$ .

The model we shall consider is an extension of the reduced BCS model used to describe ultrasmall superconducting grains [4], although our results are valid for more general cases. Let  $c_{j,\pm}^{\dagger}$  ( $c_{j,\pm}$ ) denote creation-annihilation operators for electrons in time reversal states  $|\pm\rangle$ . The index  $j = 1, \dots, N$  refers to  $N$  equally spaced energy levels  $\varepsilon_j$  with  $-\omega < \varepsilon_j < \omega$ . The energy  $\varepsilon_j$  represents the energy of a pair of electrons in a given level. The level spacing will be denoted  $2\delta$ , i.e.  $\varepsilon_{j+1} - \varepsilon_j = 2\delta$ , so that  $\omega = N\delta$  is twice the Debye energy. The energies in this paper are twice their standard values [5]. Let  $b_j = c_{j,-}c_{j,+}$ ,  $b_j^{\dagger} = c_{j,+}^{\dagger}c_{j,-}^{\dagger}$  denote the usual Cooper-pair operators. The Hilbert space  $\mathcal{H}_N$  is spanned by the combination of empty and occupied states. At half-filling the dimension of the Hilbert space is the combinatorial number  $C_{N/2}^N$ .

Our model is defined by the reduced BCS hamiltonian

$$H = \sum_{j=1}^N \varepsilon_j b_j^{\dagger} b_j - \sum_{j,j'=1}^N V_{jj'} b_j^{\dagger} b_{j'}, \quad (1)$$

where  $V_{jj'}$  is the scattering potential. In the usual BCS model,  $V_{jj'}$  is taken to be a constant. Here we add an imaginary part which breaks time reversal,

$$V_{jj'} = \begin{cases} G + i\Theta & \text{if } \varepsilon_j > \varepsilon_{j'} \\ G & \text{if } \varepsilon_j = \varepsilon_{j'} \\ G - i\Theta & \text{if } \varepsilon_j < \varepsilon_{j'} \end{cases}, \quad \begin{matrix} G = g\delta \\ \Theta = \theta\delta \end{matrix}. \quad (2)$$

This hamiltonian is hermitian since  $V_{jj'}^* = V_{j'j}$ . We consider the positive dimensionless couplings  $g$  and  $\theta$ .

The BCS variational ansatz for this model is

$$|\psi_{\text{BCS}}\rangle = \prod_{j=1}^N \left( u_j + v_j b_j^{\dagger} \right) |0\rangle. \quad (3)$$

The mean-field treatment yields the well-known equations

$$\begin{aligned} u_j^2 &= \frac{1}{2} \left( 1 + \frac{\xi_j}{E_j} \right), & v_j^2 &= \frac{1}{2} e^{2i\phi_j} \left( 1 - \frac{\xi_j}{E_j} \right), \\ E_j &= \sqrt{\xi_j^2 + \Delta_j^2}, & \xi_j &= \varepsilon_j - \mu - V_{jj}, \end{aligned} \quad (4)$$

where  $\Delta_j$  and  $\phi_j$  satisfy the gap equation:

$$\tilde{\Delta}_j = \sum_{j' \neq j} V_{jj'} \frac{\tilde{\Delta}_{j'}}{E_{j'}}, \quad \tilde{\Delta}_j \equiv \Delta_j e^{i\phi_j}. \quad (5)$$

The chemical potential equation in our case is satisfied with  $\mu = 0$ . In the thermodynamic limit,  $N \rightarrow \infty$  and  $\delta \rightarrow 0$ , with fixed  $\omega = N\delta$ , the sums over  $\varepsilon_j$  become the integrals  $\int_{-\omega}^{\omega} d\varepsilon/2\delta$ . The gap equation turns into

$$\tilde{\Delta}(\varepsilon) = g \int_{-\omega}^{\omega} \frac{d\varepsilon'}{2} \frac{\tilde{\Delta}(\varepsilon')}{E(\varepsilon')} + i\theta \left[ \int_{-\omega}^{\varepsilon} - \int_{\varepsilon}^{\omega} \right] \frac{d\varepsilon'}{2} \frac{\tilde{\Delta}(\varepsilon')}{E(\varepsilon')}, \quad (6)$$

where  $\tilde{\Delta}(\varepsilon) = \Delta(\varepsilon)e^{i\phi(\varepsilon)}$ .

Differentiating (6) with respect to  $\varepsilon$  yields

$$\frac{d\phi}{d\varepsilon} = \frac{\theta}{E(\varepsilon)}, \quad (7)$$

and the condition that  $\Delta(\varepsilon) = \Delta$  is independent of  $\varepsilon$ . The solution to eq. (7) can be taken to be

$$\phi(\varepsilon) = \theta \sinh^{-1} \frac{\varepsilon}{\Delta}. \quad (8)$$

Using eq. (8) in the gap equation (6) gives

$$1 = \int_0^{\phi(\omega)} \frac{d\phi}{\theta} (g \cos \phi + \theta \sin \phi) \implies \tan \phi(\omega) = \frac{\theta}{g}. \quad (9)$$

Solving eq. (9) for the gap yields an infinite number of solutions  $\Delta_n$ . They can be parameterized as follows:

$$\Delta_n = \frac{\omega}{\sinh t_n}, \quad t_n = t_0 + \frac{n\pi}{\theta}, \quad n = 0, 1, 2, \dots, \quad (10)$$

where  $t_0$  is the principal solution to the equation

$$\tan(\theta t_0) = \frac{\theta}{g}, \quad 0 < t_0 < \frac{\pi}{2\theta}. \quad (11)$$

The gaps satisfy  $\Delta_0 > \Delta_1 > \dots$ . Each gap  $\Delta_n$  represents a different BCS eigenstate  $|\psi_{\text{BCS}}^{(n)}\rangle$ . One can show that  $|\langle \psi_{\text{BCS}}^{(n)} | \psi_{\text{BCS}}^{(n')} \rangle| < \exp[-N(\Delta_n - \Delta_{n'})^2/8\omega^2]$ , in the limit where  $\Delta_n \ll \omega$ . Thus in the large  $N$  limit, these eigenstates are orthogonal and should all appear in the spectrum, together with the usual quasi-particle excitations above them. In the limit  $\theta \rightarrow 0$  the gaps  $\Delta_{n>0} \rightarrow 0$ , and since  $t_0 = 1/g$ ,  $\Delta_0 \sim 2\omega e^{-1/g}$ , in the weak coupling regime, recovering the standard BCS result.

For weak coupling models  $\Delta_n \ll \omega$ , all the gaps are related by a scale transformation  $\Delta_n \sim 2N\delta e^{-t_0 - n\pi/\theta}$ . Therefore, defining the condensation energy of the  $n$ -th BCS eigenstate as  $E_C^{(n)} = \langle \psi_{\text{BCS}}^{(n)} | H | \psi_{\text{BCS}}^{(n)} \rangle - E_{FS}$  we get

$$E_C^{(n)} \sim -\frac{\Delta_n^2}{8\delta} \implies E_C^{(n)} \sim -\frac{1}{2}\delta N^2 e^{-2t_0 - 2n\pi/\theta}. \quad (12)$$

Thus the spectrum of condensation energies reflects the scaling behavior of the gaps.

Next we derive RG equations for our model. Let  $g_N$ ,  $\theta_N$  denote the couplings for the hamiltonian  $H_N$  with  $N$  energy levels. The idea behind the RG method is to derive an effective hamiltonian  $H_{N-1}$  depending on renormalized couplings  $g_{N-1}$ ,  $\theta_{N-1}$  by integrating out the highest energy levels  $\varepsilon_N$  or  $\varepsilon_1$ . This can be accomplished by a canonical transformation, which is formally analogous to the one used to derive the  $t-J$  model from the Hubbard model at strong coupling [6].

We perform the calculation for general  $V$ . The integration of the level  $\varepsilon_N$  yields

$$V_{jj'}^{(N-1)} = V_{jj'}^{(N)} + \frac{1}{2} V_{jN}^{(N)} V_{Nj'}^{(N)} \left( \frac{1}{\xi_N - \xi_j} + \frac{1}{\xi_N - \xi_{j'}} \right), \quad (13)$$

where  $\xi_j = \varepsilon_j - V_{jj}$ . Integration of the level  $\varepsilon_1$  gives the same eq. (13) with the replacement  $\xi_N - \xi_j \rightarrow -\xi_1 + \xi_j$ .

Specializing to the potential eq. (2) and approximating  $\varepsilon_N - \varepsilon_j$  or  $-\varepsilon_1 + \varepsilon_j$  by  $\omega = N\delta$ , the above equation implies

$$g_{N-1} = g_N + \frac{1}{N}(g_N^2 + \theta_N^2), \quad \theta_{N-1} = \theta_N. \quad (14)$$

Thus  $\theta$  is unrenormalized.

In the large  $N$  limit one can define a variable  $s = \log N_0/N$ , where  $N_0$  is the initial size of the system. Then the beta function reads

$$\frac{dg}{ds} = (g^2 + \theta^2), \quad s \equiv \log \frac{N_0}{N}. \quad (15)$$

The solution to the above equation is

$$g(s) = \theta \tan \left[ \theta s + \tan^{-1} \left( \frac{g_0}{\theta} \right) \right], \quad g_0 = g(N_0). \quad (16)$$

The main features of this RG flow are the cyclicity

$$g(s + \lambda) = g(s) \iff g(e^{-\lambda} N) = g(N), \quad \lambda \equiv \frac{\pi}{\theta}, \quad (17)$$

and the jumps from  $+\infty$  to  $-\infty$ , when reducing the size.

The cyclicity of the RG has some important implications for the spectrum. Let  $\{E(g, \theta, N)\}$  denote the energy spectrum of the hamiltonian  $H_N$ . The RG analysis implies we can compute this spectrum using the hamiltonian  $H_{N'}(g(N'))$  if  $g(N')$  is related to  $g(N)$  according to the RG equation (16). Moreover, if  $N'$  and  $N$  are related by one RG cycle,  $N' = e^{-\lambda} N$ , then  $g(N') = g(N)$ . Thus a plot of the spectrum  $\{E(g, \theta, N)\}$  as a function of  $N$  but at fixed  $g$ ,  $\theta$  is expected to reveal the cyclicity  $\{E(g, \theta, e^{-\lambda} N)\} = \{E(g, \theta, N)\}$ . Since our RG procedure is not exact, we expect to observe this signature within the range of our approximations, i.e. for  $|E| \ll \omega$ . Indeed, this agrees with the result shown in eq. (12). This can also be observed in fig. 1 for the one Cooper pair case, with the cyclicity given by  $\lambda_1 = 2\lambda$  (see below).

Eliminating  $g_0$  in eq. (16) in terms of the mean-field solution, eqs. (10,11), we observe that the jumps in  $g(s)$  from  $+\infty$  to  $-\infty$  occur at scales  $s = t_n$ . As  $N$  decreases,

$g$  increases steadily to  $+\infty$  and then jumps to  $-\infty$ . At  $g = +\infty$ ,  $t_0 = 0$ ,  $t_1 = \pi/\theta, \dots$ , whereas for  $g = -\infty$ ,  $t_0 = \pi/\theta$ ,  $t_1 = 2\pi/\theta, \dots$ . Plugging this into eq. (10), one readily sees that

$$\begin{aligned}\Delta_0(g = +\infty) &= \infty \\ \Delta_{n+1}(g = +\infty) &= \Delta_n(g = -\infty),\end{aligned}\quad (18)$$

which indicates that at every jump the lowest condensate disappears from the spectrum, since  $E_C^{(0)}(g = +\infty) = -\infty$ . Eq. (18) implies, for the remaining condensates,  $E_C^{(n+1)}(g = +\infty) = E_C^{(n)}(g = -\infty)$ . This result is in agreement with eq. (12). Therefore, the condensate  $|\psi_{\text{BCS}}^{(n+1)}\rangle$  of one RG cycle plays the same role as  $|\psi_{\text{BCS}}^{(n)}\rangle$  of the next cycle.

The blow up of  $\Delta_0$  and  $E_C^{(0)}$  at  $g = +\infty$  is an artifact of the RG scheme used here, since we can only trust the RG for energies below the cutoff  $\omega$ . However the disappearance of bound states is correctly described by this RG (see the one Cooper pair problem for a more detailed discussion).

When  $N = \infty$  the infinite number of condensates are all expected to appear in the spectrum. However at finite  $N$  this is not possible since the Hilbert space is finite dimensional. One can use the RG to estimate the number of condensates  $n_C$  in the spectrum as a function of  $N$ . From the discussion above a condensate disappears from the spectrum for each RG cycle. Thus  $n_C$  should simply correspond to the number of cycles in  $\log N$ :

$$n_C \sim \frac{\theta}{\pi} \log N. \quad (19)$$

So far we have found a close relationship between the spectrum of our extended BCS hamiltonian in the mean-field approximation and the RG flow of the coupling constants. In order to get a further confirmation of our results we should compute the spectrum for a finite size system. However, it is very difficult to reach intermediate sizes for this model numerically, since the dimension of the Hilbert space grows as  $2^N/N^{1/2}$ . Fortunately, to this end, the similarities between the many-body case and the case of one Cooper pair, in the presence of the Fermi sea, are widely known.

For one Cooper pair in the presence of the Fermi sea, consider an eigenstate of the form  $|\psi\rangle = \sum_j \psi_j b_j^\dagger |0\rangle$ . The Schrodinger equation reads

$$(\varepsilon_j - E)\psi_j = G\psi_j + (G + i\theta) \sum_{l < j} \psi_l + (G - i\theta) \sum_{l > j} \psi_l, \quad (20)$$

with  $\varepsilon_j \in (0, \omega)$ , i.e. the Fermi sea is not accessible for the pair. Then, in the large  $N$  limit, the sums  $\sum_j$  are replaced by integrals  $\int_0^\omega d\varepsilon/2\delta$ , leading to

$$(\varepsilon - E)\psi(\varepsilon) = g \int_0^\omega \frac{d\varepsilon'}{2} \psi(\varepsilon') + i\theta \left[ \int_0^\varepsilon - \int_\varepsilon^\omega \right] \frac{d\varepsilon'}{2} \psi(\varepsilon'). \quad (21)$$

Differentiating the above with respect to  $\varepsilon$  and integrating, one obtains

$$\psi(\varepsilon) \sim \frac{1}{\varepsilon - E} e^{i\theta \log(\varepsilon - E)}. \quad (22)$$

This wave function does not have cuts in two cases, i)  $E < 0$  and ii)  $E > \omega$ . Case i) corresponds to the usual Cooper pair problem where one is looking for bound state solutions (these are the solutions we claim to have a similar behavior to the many-body case). Plugging eq. (22) back into eq. (21) one finds

$$\frac{g + i\theta}{g - i\theta} = \left(1 - \frac{\omega}{E}\right)^{i\theta}. \quad (23)$$

This equation has an infinite number of solutions given by

$$E_n = -\frac{\omega}{e^{t_n} - 1}, \quad t_n = t_0 + \frac{2\pi n}{\theta}, \quad n \in \mathbb{Z}, \quad (24)$$

where  $t_0$  is the principal solution to the equation

$$\tan\left(\frac{1}{2}\theta t_0\right) = \frac{\theta}{g}, \quad 0 < t_0 < \frac{\pi}{\theta}. \quad (25)$$

The  $n \geq 0$  solutions correspond to  $E_n < 0$ , while those with  $n < 0$  yield  $E_n > \omega$ .

As for the many-body case the spectrum has a scaling behavior for weak coupling systems, namely

$$E_n \sim -N\delta e^{-t_0 - 2n\pi/\theta}. \quad (26)$$

An RG analysis similar to the one in the many-body problem leads to the equation

$$g_{N-2} = g_N + \frac{g_N^2 + \theta_N^2}{N - 1 - g_N}, \quad \theta_{N-1} = \theta_N, \quad (27)$$

which in the large  $N$  limit becomes

$$\frac{dg}{ds} = \frac{1}{2}(g^2 + \theta^2). \quad (28)$$

The solution to this equation is given by eq. (16) just by replacing  $\theta s \rightarrow \theta s/2$ . This implies that the period of the cyclicity in  $s = \log N_0/N$  is  $\lambda_1 = 2\pi/\theta$ .

The factor 1/2 in the above formulas, as compared to the many-body problem, comes from the non accessibility of the Cooper pair to the states below the Fermi level.

The discussion leading to eq. (18) can be repeated for the one-Cooper pair problem where the role of the  $n$ -th condensate is played by the  $n$ -th bound state with energy  $E_n$  given in eq. (24), obtaining

$$\begin{aligned}E_0(g = +\infty) &= -\infty \\ E_{n+1}(g = +\infty) &= E_n(g = -\infty).\end{aligned}\quad (29)$$

Thus we expect that in each RG cycle a bound state will disappear. The analogue of equation (19) is

$$n_B \sim \frac{\theta}{2\pi} \log \frac{N}{2}, \quad (30)$$

where  $n_B$  is the number of bound states in the spectrum.

This again shows the agreement between the mean-field and the RG results. We confirm below this picture with numerical calculations.

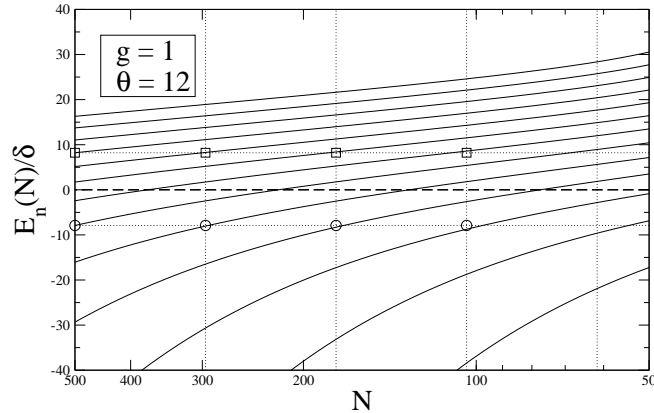


FIG. 1: Exact eigenstates of one-Cooper pair Hamiltonian for  $N$  levels, from  $N_0 = 500$  down to 50. We depict only the states nearest to zero. The vertical lines are at the values  $N_n = e^{-n\lambda_1} N_0$ . The dotted horizontal lines show the cyclicity of the spectrum.

Fig. 1 shows the numerical solution of eq. (20) for  $g = 1$ ,  $\theta = 12$  and  $N$  ranging from 500 down to 50. For each  $N$  there are  $n_B(N)$  bound states  $E_n < 0$ , where  $n_B(N)$  is in good agreement with eq. (30).

The spectrum shows the self-similarity found in the approaches above: scaling the system by a factor  $e^{-\lambda_1}$ , with  $\lambda_1 = 2\pi/\theta$ , one recovers the same spectrum for sufficiently small energies, i.e.

$$E_{n+1}(N, g, \theta) = E_n(e^{-\lambda_1} N, g, \theta). \quad (31)$$

Fig. 1 also shows the existence of critical values  $N_{c,n}$ , in the intervals  $(e^{-n\lambda_1} N, e^{-(n+1)\lambda_1} N)$ , where the bound state closest to the Fermi level disappears into the “continuum”. This effect leads to the reshuffling of bound states,  $n + 1 \rightarrow n$ , observed in eq. (31). The critical sizes are also related by scaling, i.e.  $N_{c,n}/N_{c,n+1} = e^{-\lambda_1}$ . All these phenomena are in good agreement with the RG interpretation we proposed, where the condensates disappear at scales where  $g(s = t_n) = +\infty$ . All these points are related by the scaling factor  $e^{-\lambda_1}$ .

The RG behavior is presented in fig. 2, which shows the eigenvalues  $E_n(N)$  of the one-Cooper pair Hamiltonian, with  $g_N$  running under eq. (27). The spectrum remains unchanged for  $E_n(N) \ll N\delta$ , as shown in fig. 2b. In fig. 2a one observes that for the energies  $E_n(N) \gtrsim N\delta$  the result of the RG is not reliable. Nevertheless the RG flow describes qualitatively the disappearance of the lowest bound state and the reshuffling of energy levels after a cycle, and furthermore at the predicted scales.

In summary, we have shown that adding to the standard BCS Hamiltonian a time reversal breaking term, parametrized by a coupling constant  $\theta$ , generates an infinite number of condensates with energy gaps  $\Delta_n$  related, for weak BCS couplings  $g$ , by a scale factor  $e^{-\lambda}$  with  $\lambda = \pi/\theta$ . This unusual spectrum is explained by the cyclic behavior of the RG flow of  $g$ , which reproduces itself after a finite RG time  $s$  equal to  $\lambda$ . We have also solved the finite temperature BCS gap equation, obtaining a critical temperature  $T_{c,n}$  for the  $n$ -th condensate which is related to the zero temperature gap  $\Delta_n(0)$  exactly as in the BCS theory, i.e.  $\Delta_n(0)/T_{c,n} \cong 3.52$  for weak couplings.

The simplicity of the model proposed in this letter suggests that *Russian doll superconductors* could perhaps be realized experimentally. Finally, we point out that the critical temperature can be raised by varying  $\theta$ .

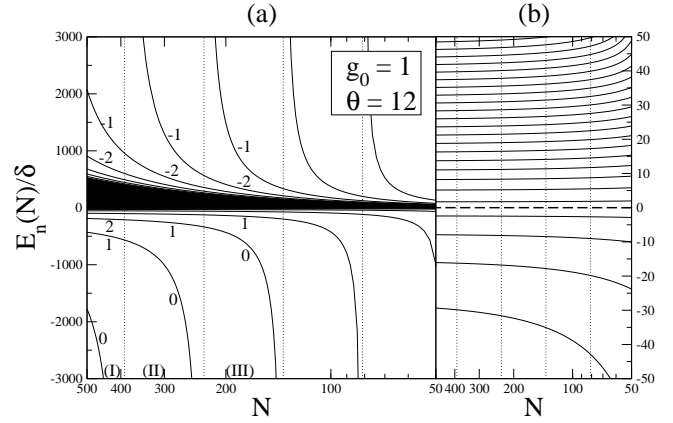


FIG. 2: Eigenstates of one-Cooper pair Hamiltonian with  $g_N$  given by eq.(27) with  $g_0 = 1$  and  $\theta = 12$ . The vertical lines denote the positions at which  $g$  jumps from  $+\infty$  to  $-\infty$ .

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